

CHAPTER-2, MATHEMATICAL LOGIC-II

Arpitika B
DMS Notes

open statements; quantifiers:

consider,

1) $x+3=6$, 2) $x^2 < 10$, 3) x divides 4 A) $x = \sqrt{2}$

according chapter 1, we know that, these sentences are not propositions unless the symbol x is specified.

This kind of sentences are called open statements, & unspecified symbols, such as x in the given sentences are called free variables.

consider, $x+3=6 \rightarrow$ open statement
 \rightarrow free variable

along with this consider set of real numbers R .

* This sentence becomes a proposition if ' x ' is replaced by any element ' R '.

eg: If $x=3$, then $3+3=6$, sentence is true proposition.

If $x=5$, then ~~$5+3=6$~~ $5+3=6$ is false proposition.

Here, ' R ' is a universe for the variable ' x ' in the sentence $x+3=6$.

* open statements containing a variable x are denoted by $p(x), q(x)$ etc.

If U is the universe for the variable x in an open statement $p(x)$ & if $a \in U$, then the proposition got by replacing x by a in $p(x)$ is denoted by $p(a)$.

* Thus if the set of all integers is the universe for x in the open statement $p(x): x+3=6$, then $p(x)$ is the proposition " $x+3=6$ " & this proposition is false.

Finally, an open statement $p(x)$ becomes a proposition only when ' x ' is replaced by a chosen element of the universe.

* The truth or falsity of the proposition $p(a)$ depends upon the element a of the universe that is chosen to replace x .

② Like compound statements or propositions, compound open statements can be formed by using the logical connectives.

(Thus, $\neg p(x)$ is negation of an open statement $p(x)$.)

also, for open statements $p(x)$ & $q(x)$,

* $p(x) \wedge q(x) \Rightarrow$ conjunction

* $p(x) \vee q(x) \Rightarrow$ disjunction

* $p(x) \rightarrow q(x) \Rightarrow$ conditional

* $p(x) \leftrightarrow q(x) \Rightarrow$ biconditional

eg ①: suppose the universe consists of all integers. consider the following open statements.

$p(x): x \leq 3$, $q(x): x+1$ is odd, $r(x): x > 0$

write down the truth values of the following:

(i) $p(2)$

Ans: $p(x): x \leq 3$

we have $p(2)$

$\therefore p(2): 2 \leq 3$, which is true

$\therefore p(2)$ is true proposition

(ii) $\neg q(4)$

Ans: $q(x): x+1$ is odd

$q(4): 4+1$

$\Rightarrow q(4): 5$ is odd

~~is odd~~ $\neg q(4): 5$ is even

$\therefore q(4)$ is false proposition

(iii) $p(-1) \wedge q(1)$ we have $p(x): x \leq 3 \wedge q(x): x+1$ is odd

$\Rightarrow p(-1): -1 \leq 3 \rightarrow$ true proposition

$q(1): 1+1 = 2$ is even \rightarrow false proposition

\therefore true \wedge false

$0 \wedge 1$

$\Rightarrow 0$

$\therefore p(-1) \wedge q(1)$ is false

$$(iv) \neg p(3) \vee r(0)$$

Soln: we have $p(x): x \leq 3$
 $r(x): x > 0$

$$\therefore p(3): 3 \leq 3 \Rightarrow \text{true} \quad r(0): 0 > 0 \Rightarrow \text{false}$$
$$\neg p(3) \Rightarrow \text{false}$$

$$\therefore p(x): x \leq 3 \vee r(x): x > 0$$
$$\text{false} \vee \text{false}$$
$$0 \vee 0$$
$$\Rightarrow \underline{\underline{0}} \text{ (false)}$$

$$(v) p(0) \rightarrow q(0)$$

Ans: $p(x): x \leq 3$

$$p(0): 0 \leq 3 \Rightarrow \underline{\underline{\text{true}}}$$

$q(x): x+1$ is odd

$$q(0): 0+1 = 1 \text{ is odd} \Rightarrow \underline{\underline{\text{true}}}$$

$$\therefore p(0) \rightarrow q(0)$$

$$1 \rightarrow 1$$

$$\Rightarrow \underline{\underline{1}} \text{ (true)}$$

$$(vi) p(1) \leftrightarrow \neg q(2)$$

$$p(1): 1 \leq 3 \Rightarrow \text{true}$$

$$\neg q(2): 2+1 \text{ is odd}$$
$$\Rightarrow \text{false}$$

$$\therefore \text{true} \leftrightarrow \text{false}$$

$$\Rightarrow \text{false}$$

$$\therefore p(1) \leftrightarrow \neg q(2) \Rightarrow \underline{\underline{\text{false}}}$$

$$(vii) p(4) \vee (q(1) \wedge r(2))$$

$$p(4): 4 \leq 3$$

$$\Rightarrow \text{false}$$

$$q(1): 1+1 \text{ is odd}$$

$$\Rightarrow \text{false}$$

$$r(2): 2 > 0$$

$$\Rightarrow \text{true}$$

$$\therefore p(4) \vee (q(1) \wedge r(2))$$
$$\text{false}$$

$$\therefore \text{false} \vee \text{false}$$

$$\Rightarrow \text{false (0)}$$

$$\therefore p(4) \vee (q(1) \wedge r(2)) \Rightarrow \underline{\underline{\text{false}}}$$

④ (viii) $p(2) \wedge (q(0) \vee \neg r(2))$

solo: $p(2): 2 \leq 3 \Rightarrow \text{true}$

$q(0): 0+1$ is odd $\Rightarrow \text{true}$

$r(2): 2 > 0 \Rightarrow \text{true}$

$\neg r(2) \Rightarrow \text{false}$

$\therefore p(2) \wedge (q(0) \vee \neg r(2))$
true

$\text{true} \wedge \text{true}$

$\Rightarrow \text{true}$

i.e., $p(2) \wedge (q(0) \vee \neg r(2)) \Rightarrow \text{true}$

Quantifiers:

consider the following propositions:

(1) All squares are rectangles

(2) For every integer x , x^2 is a non-negative integer

(3) Some determinants are equal to zero

(4) There exists a real number whose square is equal to itself.

* In these propositions the words all, every, some, there exists are associated with the idea of a quantity.

Such words are called quantifiers.

* The above propositions can be rewritten in alternative forms as follows:

Let, $S \Rightarrow$ denote set of all squares.

Then, (1) All squares are rectangles, can be written as,

For all $x \in S$, x is a rectangle.

Symbolically, written as,

$$\forall x \in S, p(x)$$

$\forall \Rightarrow$ denotes for all

$p(x) \Rightarrow$ open statement " x is a rectangle."

By,

consider proposition (2)

For every Integer x , x^2 is a non-negative Integer.

$\forall x \in \mathbb{Z}, q(x)$

↓

For every

open statement i.e., x^2 is a non-negative Integer.

i.e., the symbol ' \forall ' is used to denote the phrases

"for all", "for every", "for each" and "for any"

Hence all these phrases can be taken as equivalent phrases, each of these phrases is called the universal quantifier.

consider the proposition (3)

\Rightarrow some determinants are equal to zero.

* If 'D' denotes the set of all determinants, then proposition can be rewritten as,

For some $x \in D$, x is equal to zero.

Symbolically, $\exists x \in D, p(x)$

where, symbol \exists denotes the phrase "for some" &

$p(x)$ denotes open statement "x is equal to zero."

By, consider proposition (4)

There exists a real number whose square is equal to itself.

Symbolically written as,

$\exists x \in \mathbb{R}, q(x)$

where symbol \exists now denotes the phrase "there exists",

\mathbb{R} denotes the set of all real numbers &

$q(x)$ denotes open statement

"x is a real number whose square is equal to itself".

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i.e., the symbol \exists can be used to denote the phrases, "for some", "there exists" & "for atleast one".

Each of these equivalent phrases is called the "existential quantifier".

* A proposition involving the universal or the existential quantifier is called a quantified statement.

eg: $\forall x \in S, p(x)$

$\exists x \in S, p(x)$

* The variable present in a quantified statement is called a bound variable.

eg (2): For the universe of all Integers, let

$p(x) : x > 0$

$q(x) : x$ is even

$r(x) : x$ is a perfect square

$s(x) : x$ is divisible by 3.

$t(x) : x$ is divisible by 7.

write down the following quantified statements in symbolic form:

- (i) Atleast one Integer is even.
- (ii) There exists a positive Integer that is even
- (iii) some even Integers are divisible by 3.
- (iv) ~~For~~ Every Integer is either even or odd.
- (v) If x is even & a perfect square, then x is not divisible by 3.
- (vi) If x is odd or is not divisible by 7, then x is divisible by 3.

soln: using the definition of quantifiers, given statements read as following symbolic forms:

(i) $\exists x, q(x)$ (ii) $\exists x, [p(x) \wedge q(x)]$

(iii) $\exists x, [q(x) \wedge s(x)]$ (iv) $\forall x, [q(x) \vee \neg q(x)]$

(v) $\forall x, [q(x) \wedge r(x)] \rightarrow \neg s(x)$

(vi) $\forall x, [\neg (q(x) \vee \neg t(x))] \rightarrow s(x)$

Truth value of a quantified statement.

The following rules are employed for determining the truth value of a quantified statement.

Rule 1: The statement $\forall x \in S, p(x)$ is true only when $p(x)$ is true for each $x \in S$.

Rule 2: The statement $\exists x \in S, p(x)$ is false only when $p(x)$ is false for every $x \in S$.

Accordingly, to infer that a proposition of the form " $\forall x \in S, p(x)$ " is false, it is enough to exhibit one element a of 'S' such that $p(a)$ is false. This element 'a' is called a counter example.

* To infer that a proposition of the form " $\exists x \in S, p(x)$ " is true, it is enough to exhibit one element of S, such that $p(a)$ is true.

Two Rules of Inference:

Rule 3: If an open statement $p(x)$ is known to be true for all x in a universe 'S' and if $a \in S$, then $p(a)$ is true. known as [Rule of universal specification].

Rule 4: If an open statement $p(x)$ is proved to be true for any x chosen from a set of S, then the quantified statement $\forall x \in S, p(x)$ is true. known as (Rule of universal generalization).

Logical equivalence:

Two quantified statements are said to be logically equivalent whenever they have the same truth values in all possible situations.

- eg:
- ① $\forall x, [p(x) \wedge q(x)] \Leftrightarrow (\forall x, p(x)) \wedge (\forall x, q(x))$
 - ② $\exists x, [p(x) \vee q(x)] \Leftrightarrow (\exists x, p(x)) \vee (\exists x, q(x))$
 - ③ $\exists x, [p(x) \rightarrow q(x)] \Leftrightarrow \exists x, [\neg p(x) \vee q(x)]$

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Rule for Negation of a quantified statement:

Rule 5: To construct the negation of a quantified statement, change the quantifier from universal to existential & vice versa & also replace the open statement by its negation.

$$\text{i.e., } \neg \{ \forall x, p(x) \} \equiv \exists x, \{ \neg p(x) \}$$

$$\& \neg \{ \exists x, p(x) \} \equiv \forall x, \{ \neg p(x) \}$$

Ex (3): consider,

$$p(x): x > 0$$

$$q(x): x \text{ is even}$$

$$r(x): x \text{ is a perfect square}$$

$$s(x): x \text{ is divisible by 3,}$$

$$t(x): x \text{ is divisible by 7.}$$

Express each of the following symbolic statements in words and indicate its truth value.

sol:

(i) $\forall x, [r(x) \rightarrow p(x)]$ If $x=0$

Ans: For any integer x , if x is a perfect square, then $x > 0$.
 \Rightarrow False.

(ii) $\exists x, [s(x) \wedge \neg q(x)]$ take $x=9$

Ans: For some integer x , x is divisible by 3, and x is not even.
 \Rightarrow True.

(iii) $\forall x, [\neg r(x)]$

Ans: For any integer x , x is not a perfect square.
 \Rightarrow False

(iv) $\forall x, [r(x) \vee t(x)]$ take $x=8$

Ans: For any integer x , x is a perfect square or x is divisible by 7.

False

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4) Consider the following open statements with the set of all real numbers as the universe.

$$p(x) : |x| > 3, \quad q(x) : x > 3$$

Find the truth value of the statement

$$\forall x, [p(x) \rightarrow q(x)].$$

Also write down the converse, Inverse and the contrapositive of this statement & find their truth values.

Soln: For example take $x = -4$

$$\therefore p(-4) \equiv |-4| > 3 \equiv 4 > 3 \text{ is true}$$

$$\& q(-4) \equiv -4 > 3 \text{ is false}$$

Thus, $p(x) \rightarrow q(x)$ is false for $x = -4$.

Converse: $\forall x, [q(x) \rightarrow p(x)]$

In words: For every real number x , if $x > 3$ then $|x| > 3$
 \rightarrow True statement

Inverse: $\forall x, [\neg p(x) \rightarrow \neg q(x)]$

In words: For every real number x , if $|x| \leq 3$, then $x \leq 3$.

\rightarrow True statement [as converse & Inverse are logically equivalent].

Contrapositive: $\forall x, [\neg q(x) \rightarrow \neg p(x)]$

For every real number which is less than or equal to 3 has its magnitude less than or equal to 3.

eg (5) Consider the following open statements with the set of all real numbers as the universe.

$$p(x) : x > 0, \quad q(x) : x^2 > 0$$

$$r(x) : x^2 - 3x - 4 = 0, \quad s(x) : x^2 - 3 > 0.$$

Determine the truth values of the following statements.

Ques: (i) $\exists x, p(x) \wedge q(x)$

Soln: There exists a real number x for which both of

$p(x)$ & $q(x)$ are true, eg: $x = 1$, Therefore,

$\exists x, p(x) \wedge q(x)$ is true statement, truth value = 1 //

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(ii) $\forall x, p(x) \rightarrow q(x)$

For any real number $q(x)$ is true,
Hence $p(x) \rightarrow q(x)$ cannot be false for any real x .
 $\therefore \forall x, p(x) \rightarrow q(x)$ is true. Truth value = 1

(iii) $\forall x, q(x) \rightarrow s(x)$

Ans: We can note the $s(x)$ is false for $x=1$
& $q(x)$ is true for $x=1$.

Thus, $q(x) \rightarrow s(x)$ is false for $x=1$,

$\therefore \forall x, q(x) \rightarrow s(x)$ is false. Truth value = 0

(iv) $\forall x, r(x) \vee s(x)$

Ans: we have, $x^2 - 3x - 4 = 0$

$$x^2 - 4x + x - 4 = 0$$

$$x(x-4) + 1(x-4) = 0$$

$$(x-4)(x+1) = 0$$

$$\Rightarrow x^2 - 3x - 4 = (x-4)(x+1)$$

$\therefore r(x)$ is true only for $x=4$ & $x=-1$.

$\therefore r(x)$ & $s(x)$ false for $x=1$.

such that $r(x) \vee s(x)$ is not always true.

$\therefore \forall x, r(x) \vee s(x)$ is false truth value = 0

(v) $\exists x, p(x) \wedge r(x)$

Ans: From the previous example, we note that
 ~~$x=4$~~ $r(x)$ is true only for $x=4$ or $x=-1$

$\therefore p(x)$ and $r(x)$ are true for $x=4$,

$\therefore \exists x p(x) \wedge r(x)$ is true.

Truth value = 1

(vi) $\forall x, r(x) \rightarrow p(x)$

$r(x)$ true for $x=4$ or $x=-1$

\therefore we can observe, $p(x)$ is false for $x=-1$ &
 $r(x)$ is true for $x=-1$.

$\therefore r(x) \rightarrow p(x)$ is false for $x=-1$ //.

$\therefore r(x) \rightarrow p(x)$ not always true

(ii)

Eq ⑥ Let $p(x): x^2 - 7x + 10 \geq 0$ $q(x): x^2 - 2x - 3 \geq 0$, $r(x): x < 0$
 Determine the truth or falsity of the following statements when the universe U contains only the integers 2 and 5. If a statement is false, provide a counter example or explanation.

- (i) $\forall x, p(x) \rightarrow \neg r(x)$ (ii) $\forall x, q(x) \rightarrow r(x)$
 (iii) $\exists x, q(x) \rightarrow r(x)$ (iv) $\exists x, p(x) \rightarrow r(x)$

Soln: Here, the universe is $U = \{2, 5\}$.

We have, $p(x): x^2 - 7x + 10 \geq 0$
 $x^2 - 5x - 2x + 10 \geq 0$
 $\equiv (x-5)(x-2)$
 $\equiv (x=5) \text{ and } x=2$

$\therefore p(x)$ is true for $x=5$ & $x=2$
 i.e., as $U = \{2, 5\}$, $p(x)$ is true for all $x \in U$.

$q(x): x^2 - 2x - 3 \geq 0$
 $x^2 - 3x + 1x - 3 \geq 0$
 $(x-3)(x+1)$
 $\equiv x=3 \text{ and } x=-1$

$\therefore q(x)$ is true for $x=3$ & $x=-1$
 But 3 & -1 are not in $U = \{2, 5\}$
 $\therefore q(x)$ is false for all $x \in U$.

as $U = \{2, 5\}$, obviously, $r(x): x < 0$ is false for all $x \in U$.

Accordingly:

(i) $\forall x, p(x) \rightarrow \neg r(x)$

Since $p(x)$ is true for all $x \in U$ & $\neg r(x)$ is true for all $x \in U$,

$\therefore \forall x, p(x) \rightarrow \neg r(x)$ is true.

(ii) Since $q(x)$ is false for all $x \in U$ & $r(x)$ is false for all $x \in U$, the statement
 $\forall x, q(x) \rightarrow r(x)$ is true.

(12) (iii) $\exists x, q(x) \rightarrow r(x)$
 since $q(x)$ & $r(x)$ are false for $x=2$, the
 statement $\exists x, q(x) \rightarrow r(x)$ is true

(iv) since $p(x)$ is true for all $x \in U$ but $r(x)$ is
 false for all $x \in U$, the statement $p(x) \rightarrow r(x)$
 is false for every $x \in U$.
 consequently, $\exists x, p(x) \rightarrow r(x)$ is false.

eg 7: Negate & simplify each of the following:

(i) $\exists x, [p(x) \vee q(x)]$
 $\neg [\exists x, [p(x) \vee q(x)]]$
 $\equiv \forall x [\neg \{p(x) \vee q(x)\}]$
 $\equiv \forall x [\neg p(x) \wedge \neg q(x)]$

(ii) $\forall x, [p(x) \rightarrow q(x)]$
 $\equiv \neg \{ \forall x, [p(x) \rightarrow q(x)] \}$
 $\equiv \exists x, [\neg \{p(x) \rightarrow q(x)\}]$
 $\equiv \exists x, [p(x) \wedge \neg q(x)]$

(iii) $\forall x, [p(x) \wedge \neg q(x)]$
 soln: $\neg \{ \forall x, [p(x) \wedge \neg q(x)] \}$
 $\exists x, [\neg \{p(x) \wedge \neg q(x)\}]$
 $\exists x, [\neg p(x) \vee q(x)]$

(iv) $\exists x, [\{p(x) \vee q(x)\} \rightarrow r(x)]$
 $\neg \{ \exists x, [\{p(x) \vee q(x)\} \rightarrow r(x)] \}$
 $\forall x, [\neg \{ \{p(x) \vee q(x)\} \rightarrow r(x) \}]$
 $\forall x, [\{p(x) \vee q(x)\} \wedge \neg r(x)]$

eg 8: Let the set Z of all integers be the universe. consider
 the statements

$p(x): 2x+1=5$ & $q(x): x^2=9$.

obtain the negation of the quantified statement.

$\exists x \in Z, [p(x) \wedge q(x)]$ & express it in words.

soln: $\neg \{ \exists x \in Z, [p(x) \wedge q(x)] \}$
 $\forall x \in Z, [\neg \{p(x) \wedge q(x)\}]$
 $\forall x \in Z, [\neg p(x) \vee \neg q(x)]$

In words: For all integers x , $2x+1 \neq 5$ or $x^2 \neq 9$

(13)

Ex 9: Let the universe comprise of all integers. Given
 $p(x): x$ is odd & $q(x): x^2 - 1$ is even,

Express the conditional

for any x , if x is odd, then $x^2 - 1$ is even

in symbolic form & negate it.

Soln:

$$\forall x \in \mathbb{Z}, [p(x) \rightarrow q(x)]$$

Let Z denote the set of all integers. Then, in symbolic form, the given conditional reads:

$$\forall x \in \mathbb{Z}, [p(x) \rightarrow q(x)]$$

Negation: $\neg [\forall x \in \mathbb{Z}, [p(x) \rightarrow q(x)]]$

$$\exists x \in \mathbb{Z}, [\neg [p(x) \rightarrow q(x)]]$$

$$\exists x \in \mathbb{Z}, [p(x) \wedge \neg q(x)]$$

In words:

For some integer x , x is odd and $x^2 - 1$ is not even

Ex 10: Write down the following proposition in symbolic form & find its negation.

"All integers are rational numbers and some rational numbers are not integers"

Soln:

Let $p(x): x$ is rational number, $q(x): x$ is an integer

and Z : set of all integers. Q : set of all rational numbers.

Then, in symbolic form, the given proposition reads,

$$[\forall x \in Z, p(x)] \wedge [\exists x \in Q, \neg q(x)]$$

Negation of this:

$$\neg [\forall x \in Z, p(x)] \vee \neg [\exists x \in Q, \neg q(x)]$$

$$\equiv [\exists x \in Z, \neg p(x)] \vee [\forall x \in Q, q(x)]$$

In words: "Some integers are not rational numbers or every rational number is an integer"

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Q(11):

write down the following proposition in symbolic form, & find its negation:

"If all triangles are right-angled, then no triangle is ~~squarelike~~ equiangular."

Sols: Let T denotes the set of all triangles. Also let,

$p(x)$: x is right-angled, $q(x)$: x is equiangular

Then, in symbolic form the given proposition reads,

$$\{\forall x \in T, p(x)\} \rightarrow \{\forall x \in T, \neg q(x)\}$$

negation: $\{\forall x \in T, p(x)\} \wedge \{\exists x \in T, q(x)\}$

words: "All triangle are right-angled & some triangles are equiangular".

Q(12): write down the negation of each of the following statements.

(i) For all integers n , if n is not divisible by 2, then n is odd.

(ii) If k, m, n are any integers where $(k-m)$ and $(m-n)$ are odd, then $(k-n)$ is even.

(iii) For all real numbers x , if $|x-3| < 7$, then $-4 < x < 10$.

(iv) If x is a real number where $x^2 > 16$, then $x < -4$ or $x > 4$.

Sols: let Z denote the set of all integers and R denote the set of all real number. Then,

(i) The given statement is,

$$\forall n \in Z, \neg p(n) \rightarrow q(n)$$

where, $p(n)$: n is divisible by 2, $q(n)$: n is odd.

Therefore, the negation of the given statement is,

$$\exists n \in Z, \neg p(n) \wedge \neg q(n)$$

In words:

For some integer n , n is not divisible by 2 and n is not odd.

(ii)

$$\forall k, m, n \in \mathbb{Z}, [p(x) \wedge q(x)] \rightarrow r(x),$$

where $p(x): k-m$ is odd, $q(x): m-n$ is odd, $r(x): k-n$ is even.

The negation of this is,

$$\exists k, m, n \in \mathbb{Z}, [p(x) \wedge q(x)] \wedge \neg r(x)$$

In words,

There exist integers k, m, n such that $k-m, m-n$ are odd and $k-n$ is not even

(iii) $\forall x \in \mathbb{R}, p(x) \rightarrow q(x),$

where, $p(x): |x-3| < 7, q(x): -4 < x < 10$; i.e, $x \in (-4, 10)$

negation: $\exists x \in \mathbb{R}, p(x) \wedge \neg q(x)$; i.e,

for some real number $x, |x-3| < 7$ and $x \notin (-4, 10)$

(iv) $\forall x \in \mathbb{R}, p(x) \rightarrow q(x) \vee r(x)$

where, $p(x): x^2 > 16, q(x): x < -4, r(x): x > 4$.

negation: $\exists x \in \mathbb{R}, [p(x) \wedge \neg q(x) \wedge \neg r(x)]$;

i.e, for some real number $x, x^2 > 16$ and $x \geq -4$ and $x \leq 4$

Logical Implication Involving Quantifiers:

* A quantified statement 'p' is said to be logically imply a quantified statement Q, if Q is true whenever 'p' is true.

Then, $P \Rightarrow Q$.

consider a set of quantified statements P_1, P_2, \dots, P_n and Q we say that Q is valid conclusion from the premises P_1, P_2, \dots, P_n or that

$$P_1 \wedge P_2 \wedge \dots \wedge P_n \rightarrow Q$$

is a valid argument if Q is true whenever each of P_1, P_2, \dots, P_n is true or equivalently if

$$P_1 \wedge P_2 \wedge \dots \wedge P_n \Rightarrow Q.$$

* Note: The validity of an argument involving quantified statements is analysed on the basis of the laws of logic & the rules of Inference of 1st chapter.

eg: (i) prove the following:

$$(i) \forall x, p(x) \Rightarrow \exists x, p(x)$$

(let 'S' denote universe.)

$$\forall x, p(x) \Rightarrow p(x) \text{ is true for all } x \in S$$

$$\Rightarrow p(a) \text{ is true for } x=a \in S$$

$$\Rightarrow p(x) \text{ is true for some } x \in S$$

$$\Rightarrow \underline{\underline{\exists x, p(x)}}$$

$$(ii) \forall x, [p(x) \vee q(x)] \Rightarrow \forall x, p(x) \vee \exists x, q(x)$$

$$\Rightarrow \forall x, [p(x) \vee q(x)]$$

$$\Rightarrow \{ \forall x, p(x) \text{ is true for every } x \in S \} \vee \{ q(x) \text{ is true for every } x \in S \}$$

$$\Rightarrow \{ \forall x, p(x) \} \vee q(a) \text{ is true for } x=a \in S$$

$$\Rightarrow \forall x, p(x) \vee q(x) \text{ is true for some } x \in S$$

$$\Rightarrow \underline{\underline{\forall x, p(x) \vee \exists x, q(x)}}$$

2) prove that,

$\exists x, [p(x) \wedge q(x)] \Rightarrow \exists x p(x) \wedge \exists x, q(x)$ is the converse true?

Soln: Let 'S' denote the universe, we find that,

$$\exists x, [p(x) \wedge q(x)]$$

$$\Rightarrow p(a) \wedge q(a) \text{ for some } a \in S$$

$$\Rightarrow p(a), \text{ for some } a \in S \text{ and } q(a), \text{ for some } a \in S$$

$$\Rightarrow \exists x, p(x) \wedge \exists x, q(x)$$

This proves required Implication.

Now let, $\exists x, p(x) \Rightarrow p(a)$ for some $a \in S$ and

$\exists x, q(x) \Rightarrow q(b)$ for some $b \in S$.

$$\therefore \exists x, p(x) \wedge \exists x, q(x) \Rightarrow p(a) \wedge q(b)$$

$$\not\leftrightarrow p(a) \wedge q(a)$$

Because b need not be same as a.

$\therefore \exists x, [p(x) \wedge q(x)]$ need not be true when,

$\exists x, p(x) \wedge \exists x, q(x)$ is true.

$$\text{i.e., } \exists x, p(x) \wedge \exists x, q(x) \not\leftrightarrow \exists x, [p(x) \wedge q(x)]$$

Accordingly the converse of the given Implication is not necessarily true.

$$\therefore \exists x, [p(x) \wedge q(x)] \not\leftrightarrow \exists x, p(x) \wedge \exists x, q(x)$$

3) prove that the statement $\exists x, q(x)$ follows logically from the premises $\forall x, p(x) \rightarrow q(x)$ and $\exists x, p(x)$

Soln: we have,

$$[\exists x, p(x)] \wedge [\forall x, p(x) \rightarrow q(x)]$$

we know, $\exists x p(x) \Rightarrow p(a)$ for some a in the universe.

$$\therefore [\exists x p(x)] \wedge [\forall x, p(x) \rightarrow q(x)]$$

$$\Rightarrow p(a) \wedge [p(a) \rightarrow q(a)]$$

$$\Rightarrow q(a) \quad [\text{modus ponens rule}]$$

$$\Rightarrow \exists x, q(x) \quad //$$

4) prove that the following argument is valid.

All men are mortal

Sachin is a man

\therefore Sachin is mortal

Soln: $S \rightarrow$ denote the set of all men.

$p(x) \rightarrow$ denote statement 'x is mortal'

$a \rightarrow$ denote Sachin.

$\therefore \forall x \in S, p(x)$

$a \in S$

$\therefore p(a)$

Since $\forall x \in S, p(x)$ is true, & $a \in S$, by rule of universal specification $p(a)$ is true.

Thus given argument is valid

5) Find whether the following argument is valid.

No engineering student of first or second semester studies logic.

Anil is an engineering student who studies logic

\therefore Anil is not in second semester.

Soln: Universe: set of all engineering students

$p(x)$: x is in first sem. $q(x)$: x is in second sem

$r(x)$: x studies logic. a : Anil.

Given argument:

$\forall x, [\{p(x) \vee q(x)\} \rightarrow \neg r(x)]$

$\therefore r(a)$

$\therefore \neg q(a)$

$\{\forall x, [\{p(x) \vee q(x)\} \rightarrow \neg r(x)]\}$

$\Rightarrow \{p(a) \vee q(a)\} \rightarrow \neg r(a)$ By the rule of universal specification

Therefore,

- $\{\forall x, [p(x) \vee q(x) \rightarrow \neg r(x)] \wedge r(a)\}$
 - $\Rightarrow [p(a) \vee q(a) \rightarrow \neg r(a)] \wedge r(a)$
 - $\Rightarrow [r(a) \rightarrow \neg [p(a) \vee q(a)]] \wedge r(a)$ contrapositive
 - $\Rightarrow r(a) \wedge [r(a) \rightarrow \neg [p(a) \vee q(a)]]$ commutative law.
 - $\Rightarrow \neg [p(a) \vee q(a)]$ modus ponens rule.
 - $\Rightarrow \neg p(a) \wedge \neg q(a)$ De Morgan law
 - $\Rightarrow \underline{\neg q(a)}$, rule of conjunctive simplification
- Hence given argument is true

6) prove the following argument is valid.

$$\begin{array}{l} \forall x, [p(x) \rightarrow q(x)] \\ \forall x, [q(x) \rightarrow r(x)] \\ \hline \therefore \forall x, [p(x) \rightarrow r(x)] \end{array}$$

Soln: Take any a from the universe, then

- $\{\forall x, [p(x) \rightarrow q(x)]\} \wedge \{\forall x, [q(x) \rightarrow r(x)]\}$
- $\Rightarrow \{p(a) \rightarrow q(a)\} \wedge \{q(a) \rightarrow r(a)\}$
- $\Rightarrow p(a) \rightarrow r(a)$, rule of syllogism.
- $\Rightarrow \underline{\forall x, [p(x) \rightarrow r(x)]}$, rule of universal generalization

given argument is valid

7) prove that the following argument is valid.

$$\begin{array}{l} \forall x, [p(x) \rightarrow \{q(x) \wedge r(x)\}] \\ \forall x, [p(x) \wedge s(x)] \\ \hline \therefore \forall x, [r(x) \wedge s(x)] \end{array}$$

Take any x from the universe,
Then,

$$[p(x) \rightarrow \{q(x) \wedge r(x)\}] \wedge [p(x) \wedge s(x)]$$

$\Leftrightarrow p(x) \wedge [p(x) \rightarrow \{q(x) \wedge r(x)\}] \wedge s(x)$ By associative law & commutative law.

$\Rightarrow \{q(x) \wedge r(x)\} \wedge s(x)$ modus ponens rule

$\Rightarrow q(x) \wedge \{r(x) \wedge s(x)\}$ Associative law

$\Rightarrow \{r(x) \wedge s(x)\}$ rule of conjunctive simplification

In view of rule of universal generalization, the given argument is valid

eg ⑧ Establish the validity of the following argument.

If a triangle has two equal sides, then it is isosceles

If a triangle is isosceles, then it has two equal angles

A certain triangle ABC does not have two equal angles

\therefore The triangle ABC does not have two equal sides

Sol: Let the universe be set of all triangles &

$\Rightarrow p(x)$: x has equal sides

$q(x)$: x is isosceles

$r(x)$: x has two angles

Also, let c denote the triangle ABC.

Then, in symbols, given argument reads as follows:

$$\forall x, [p(x) \rightarrow q(x)]$$

$$\forall x, [q(x) \rightarrow r(x)]$$

$$\neg r(c)$$

$$\therefore \neg p(c)$$

$$\forall x, [p(x) \rightarrow q(x)] \wedge [q(x) \rightarrow r(x)] \wedge \neg r(c)$$

$\Rightarrow \{\forall x, [p(x) \rightarrow r(x)]\} \wedge \neg r(c)$ rule of syllogism

$\Rightarrow \{p(c) \rightarrow r(c)\} \wedge \neg r(c)$ rule of universal specification

$\Rightarrow \neg p(c)$ modus tollens rule

This proves the given argument is valid

21) prove that the following argument is valid.

$$\begin{array}{l} \forall x, [p(x) \vee q(x)] \\ \exists x, \neg p(x) \\ \forall x, [\neg q(x) \vee r(x)] \\ \forall x, [s(x) \rightarrow \neg r(x)] \\ \hline \therefore \exists x, \neg s(x) \end{array}$$

$$\{ \forall x, [p(x) \vee q(x)] \} \wedge \{ \exists x, \neg p(x) \}$$

$\Rightarrow \{ p(a) \vee q(a) \} \wedge \{ \neg p(a) \}$ for some a in the universe

$\Rightarrow q(a)$, Disjunctive syllogism.

$$\{ \forall x, [p(x) \vee q(x)] \} \wedge \{ \exists x, \neg p(x) \} \wedge \{ \forall x, [\neg q(x) \vee r(x)] \}$$

$$\Rightarrow q(a) \wedge [\neg q(a) \vee r(a)]$$

$\Rightarrow r(a)$, By rule of disjunctive syllogism

$$\{ \forall x, [p(x) \vee q(x)] \} \wedge \{ \exists x, \neg p(x) \} \wedge \{ \forall x, [\neg q(x) \vee r(x)] \} \wedge \{ \forall x, [s(x) \rightarrow \neg r(x)] \}$$

$$\Rightarrow r(a) \wedge [s(a) \rightarrow \neg r(a)]$$

$\Rightarrow r(a) \wedge [r(a) \rightarrow \neg s(a)]$ contrapositive

$\Rightarrow \neg s(a)$ modus ponens rule

$\Rightarrow \exists x, \neg s(x)$ universal generalization

Thus proves that the given argument is valid

Statements with more than one variable:

consider the following statements:

1) $x - 2y$ is a positive Integer

2) $x + y - z = 0$

These open statements contains more than one free-variable.
* These becomes propositions if each variable is replaced by an element of a certain universe.

eg: consider set of all Integers as the universe.

Replace x & y by 5 & -3 respectively.

Then, $5 - 2(-3) = 5 - (-6)$

$= 1$ is a positive Integer \Rightarrow True.

Uy, Take set of all Integers as the universe.

where $x = 2, y = 1, z = -3$

Then, $x + y - z = 0$

$2 + 1 - 3 = 0 \Rightarrow 0 = 0$ True.

If $x = 1, y = 1, z = 3$

$1 + 1 - 3 = 0 \Rightarrow$ False.

open statements containing two variables x & y are denoted by $p(x, y), q(x, y)$ etc.

* If open statements contain 3 variables x, y & z .

Then, it is denoted by $p(x, y, z), q(x, y, z)$ etc.

* Here, the universe can be the same for all variables, or can be different for different variables.

eg: (i) Let $p(x, y)$ & $q(x, y)$ denote the following open statements:

$p(x, y): x^2 \geq y$, $q(x, y): (x+2) < y$

If the universe for both of x, y is the set of all real numbers, and determines the truth value of each of the following statements.

(i) $p(2,4)$

soln: we have,

$p(x,y) : x^2 > y$

$p(2,4) : 2^2 > 4$

$\Rightarrow 4 > 4 \Rightarrow \underline{\underline{\text{True}}}$

(ii) $q(1,\text{F})$

soln we have

$q(x,y) : (x+2) < y$

$q(1,\text{F}) : (1+2) < \text{F}$

$: 3 < \text{F} \Rightarrow \underline{\underline{\text{True}}}$

(iii) $p(-3,8) \wedge q(1,3)$

$(x^2 > y) \wedge (x+2) < y$

$[(-3)^2 > 8] \wedge (1+2) < 3$

$9 > 8 \wedge 3 < 3$

True False

$\Rightarrow \underline{\underline{\text{False}}}$

(iv) $p(1/2, 1/3) \vee \neg q(-2, -3)$

$(x^2 > y) \vee \neg (x+2) < y$

$[(1/2)^2 > (1/3)^2] \vee \neg (-2+2) < -3$

$(1/4 > 1/9) \vee \neg (0 < -3)$

False \vee True

$\Rightarrow \underline{\underline{\text{True}}}$

v) $p(2,2) \leftrightarrow q(1,1)$

$(x^2 > y) \leftrightarrow (x+2) < y$

$(2^2 > 2) \leftrightarrow (1+2) < 1$

$4 > 2 \leftrightarrow 3 < 1$

True \leftrightarrow False

$\Rightarrow \underline{\underline{\text{False}}}$

(vi) $p(1,2) \leftrightarrow \neg q(3,8)$

$(x^2 > y) \leftrightarrow \neg (x+2) < y$

$(1^2 > 2) \leftrightarrow \neg (3+2) < 8$

False $\leftrightarrow \neg$ True

False \leftrightarrow False

$\Rightarrow \underline{\underline{\text{True}}}$

Quantified statements with more than one variable

eg: ① $\forall x, \forall y, p(x,y)$

② $\exists x, \exists y, p(x,y)$

③ $\forall x, \exists y, p(x,y)$

④ $\exists x, \forall y, p(x,y)$

eg ②: Let x and y denote Integers. consider the statement.
 $p(x,y) : x+y$ is even.

write down the following statements in words:

(i) $\forall x, \exists y, p(x,y)$

Ans: with every Integer x , there exists an Integer y such that $x+y$ is even.

(ii) $\exists x, \forall y, p(x,y)$

Ans: There exists an Integer x such that $x+y$ is even for every Integer y .

③ write down the following statements in symbolic form using quantifiers:

① Every real number has an additive inverse.

Ans: "Given any real number x , there exist a real number y such that $x+y=y+x=0$.

i.e, $\forall x, \exists y, [x+y=y+x=0]$
 Here, set of all real numbers is the universe.

② The set of real numbers has a multiplicative identity.

Ans: "There exists a real number x such that $xy=yx=y$ for every y ."
 $\exists x, \forall y, [xy=yx=y]$.

Set of all real numbers is the universe.

③ The integer 58 is equal to the sum of two perfect squares.

Ans: There exist integers m and n such that $58=m^2+n^2$

$\exists m, \exists n, 58 = m^2 + n^2$

set of all integers is the universe

eg (A): Determine the truth value of each of the following quantified statements, the universe being the set of all non-zero integers.

(i) $\exists x, \exists y, [xy=1]$

soln: Take $x=1, y=1$

$\therefore \exists x, \exists y, [xy=1]$ is true

(ii) $\exists x, \forall y, [xy=1]$

False. For specified x , $xy=1$ but for every y it is not true

(iii) $\forall x, \exists y, [xy=1]$

False. For $x=2$, there is no integer y such that $xy=1$.

(iv) ~~True~~ $\exists x, \exists y [(2x+y=5) \wedge (x-3y=-8)]$

Take $x=1, y=3$

True

(v) $\exists x, \exists y, [(3x - y = 17) \wedge (2x + 4y = 35)]$

Soln: False. These two do not have common integer solution

Q(5): Find the negation of the following quantified statement:

$\forall x, \exists y, \{ [p(x, y) \wedge q(x, y)] \rightarrow r(x, y) \}$

Soln: $\neg [\forall x, \exists y, \{ [p(x, y) \wedge q(x, y)] \rightarrow r(x, y) \}]$

$\Leftrightarrow \exists x, \neg \{ \exists y, \{ [p(x, y) \wedge q(x, y)] \rightarrow r(x, y) \} \}$

$\Leftrightarrow \exists x, \forall y, \neg [p(x, y) \wedge q(x, y) \rightarrow r(x, y)]$

$\Leftrightarrow \exists x, \forall y, [p(x, y) \wedge q(x, y) \wedge \neg r(x, y)]$

Q(6): Find the negation of each of the following quantified statements.

(i) $\forall x, \forall y, [(x > y) \rightarrow ((x - y) > 0)]$

Ans: $\neg \{ \forall x, \forall y, [(x > y) \rightarrow ((x - y) > 0)] \}$

$\exists x, \exists y, [(x > y) \wedge ((x - y) \leq 0)]$

(ii) $\forall x, \forall y, [(x < y) \rightarrow \exists z, (x < z < y)]$

Ans: $\exists x \exists y, \neg [(x < y) \rightarrow \exists z, (x < z < y)]$

$\exists x, \exists y, [(x < y) \wedge \forall z, \neg (x < z < y)]$

$\exists x, \exists y, [(x < y) \wedge \forall z, \{ (x > z) \vee (z > y) \}]$

(iii) $\forall x, \forall y, [(|x| = |y|) \rightarrow (y = \pm x)]$

Ans: $\exists x, \exists y, [(|x| = |y|) \wedge (y \neq \pm x)]$

(iv) $[\forall x, \forall y, ((x < 0) \wedge (y > 0)) \rightarrow [\exists z, (x < z < y)]]$

Ans: $\exists x, \exists y, ((x < 0) \wedge (y > 0)) \wedge \forall z, (x < z < y)$

Q(7): prove that the following argument is valid, where a & b are some particular members of the universe.

$\forall x, \forall y, [p(x, y) \rightarrow q(x, y)]$
 $\neg q(a, b)$

$\therefore \exists x, \exists y, \{ \neg p(x, y) \}$

$$\{\forall x, \forall y, [p(x, y) \rightarrow q(x, y)]\} \wedge \neg q(a, b)$$

$$\Rightarrow \{p(a, b) \rightarrow q(a, b)\} \wedge \neg q(a, b)$$

$$\Rightarrow \neg p(a, b) \text{ By modus tollens rules}$$

$$\Rightarrow \exists x, \exists y, \{\neg p(x, y)\}$$

This proves that the given argument is valid

Methods of proof and methods of disproof:

- * consider the conditional $p \rightarrow q$, where p and q are simple compound propositions which may involve quantifiers.
- * Given such conditional, the process of establishing that the conditional is true by using laws of logic and other known facts constitutes a proof of the conditional.
- * The process of establishing that a proposition is false constitutes a disproof.

Direct proof:

The direct method of proving a conditional $p \rightarrow q$ has the following lines of argument:

1. Hypothesis: First assume that 'p' is true.
2. Analysis: Starting with the hypothesis & employing rules of logic & other known facts, infer that q is true.
3. Conclusion: $p \rightarrow q$ is true.

Ex: (i) Give a direct proof of the statement:

"The square of an odd integer is an odd integer"

Soln: consider the conditional,

"If n is an odd integer then n^2 is an odd integer"

Hypothesis: Assume 'n' is an odd integer

Then, $n = 2k + 1$ for some integer k
 consequently, $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$ } Analysis

we can observe right-hand side is not divisible 2 } analysis
 $\therefore n^2$ is not divisible by 2.
 $\therefore n^2$ is an odd integer (conclusion).
 \therefore The given statement is proved by a direct proof.

eg ②: prove that, for all integers k and l , if k and l are both odd, then $k+l$ is even and kl is odd.

Sol: given statement:

If k and l are odd then $k+l$ is even and kl is odd.

Take any two integers k and l , and assume both of these are odd (hypothesis).

Analysis: $k=2m+1$, $l=2n+1$ for some integers m & n .

$$\begin{aligned} \therefore k+l &= (2m+1) + (2n+1) \\ &= 2m+2n+2 \\ &= 2(m+n+1) \end{aligned}$$

$$\begin{aligned} kl &= (2m+1)(2n+1) \\ &= 4mn + 2m + 2n + 1 \end{aligned}$$

we observe that $k+l$ is divisible by 2 and kl is not divisible by 2.

$\therefore k+l$ is even integer & kl is an odd integer
(conclusion)

Indirect proof:

* we have conditional $p \rightarrow q$ & its contrapositive $\neg q \rightarrow \neg p$ are logically equivalent.

* In some situations given a conditional $p \rightarrow q$, a direct proof of the contrapositive $\neg q \rightarrow \neg p$ is easier.

on the basis of this proof, we infer that the conditional $p \rightarrow q$ is true.

This method of proving a conditional is called an indirect proof //

Q(3): Let 'n' be an Integer. prove that if n^2 is odd then 'n' is odd.

Soln: Here the conditional to be proved is $p \rightarrow q$,
where $p: n^2$ is odd and $q: n$ is odd.

First prove that the contrapositive $\neg q \rightarrow \neg p$ is true.

Assume that $\neg q$ is true.

Assume that $\neg q$ is true. i.e, assume that 'n' is not an odd Integer.

Then $n = 2k$ where k is an Integer.

consequently $n^2 = (2k)^2 = 2(2k^2)$ so that n^2 is not odd.

i.e, p is false i.e, $\neg p$ is true.

This proves contrapositive $\neg q \rightarrow \neg p$.

which is equivalent to $p \rightarrow q$

(A) give an indirect proof of the statement.

"The product of two even Integers is an even Integer"

Soln: It can be written as:

"If a & b are even Integers, then ab is an even Integer"

This conditional to be proved is $p \rightarrow q$.

where, $p: a$ & b are even Integers

$q: ab$ is an even Integer.

First prove that the contrapositive $\neg q \rightarrow \neg p$ is true.

Assume $\neg q$ is true. i.e, assume that ab is not an even Integer. i.e, ab is not divisible by 2.

Hence a is not divisible by 2 and b is not divisible by 2.

i.e, a is not^{an} even Integer and b is not an even Integer.

This means the proposition "a and b are even Integers" is false.

i.e, p is false, or $\neg p$ is true.

This proves contrapositive $\neg q \rightarrow \neg p$.

∴ It follows given statement $p \rightarrow q$ is true.

This completes the indirect proof //

Ex 5:

Provide an Indirect proof of the given statement.
 "For all positive real numbers x and y , if the product xy exceeds 25, then $x > 5$ or $y > 5$ ".

Soln: Let x and y be any two positive real numbers.

Then the given statement reads as, $p \rightarrow (q \vee r)$.

where, $p: xy$, $q: x > 5$, $r: y > 5$.

Contrapositive, $(\neg q \wedge \neg r) \rightarrow \neg p$.

Now, prove that this contrapositive is true.

Suppose, $(\neg q \wedge \neg r)$ is true, then $\neg q$ is true & $\neg r$ is true,

i.e., $x \leq 5$ & $y \leq 5$.

This gives $xy \leq 25$. So that $\neg p$ is true.

This contrapositive is true.

This proof of the contrapositive serves as an Indirect proof of the statement $p \rightarrow (q \vee r)$.

* By rule of universal generalization, this Indirect proof of $p \rightarrow (q \vee r)$ establishes truth of the given statement.

Ex 6: For each of the following statements, provide an Indirect proof by stating & proving the contrapositive of the given statement.

(i) For all integers k and l , if kl is odd, then both k and l are odd.

Soln: Contrapositive: "For all integers k and l , if k is even or l is even, then kl is even."

Prove the contrapositive:

For any integers k and l , assume that k is even.

Then $k = 2m$ for some integer m , and $kl = (2m)l = 2(ml)$

which is even. Similarly, if l is even, then $kl = k(2n)$

$= 2(kn)$ for some integer n so that kl is even.

This proves contrapositive, which is Indirect proof //

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(ii) For all integers k and l , if $k+l$ is even, then k and l are both even or both odd.

Soln: Here, the contrapositive: "For all integers k and l , if one of k and l is odd and the other is even, then $k+l$ is odd".

prove the contrapositive:

For any integers k and l , assume that, one of k and l is odd & other is even.

Suppose k is odd & l is even.

Then $k=2m+1$ and $l=2n$ for some integers m and n .

consequently $k+l=(2m+1)+2n=2(m+n)+1$, which is odd.

By if k is even and l is odd, then we find that $k+l$ is odd. This proves the contrapositive.

which is a Indirect proof

Ex (7): Let m and n be integers. prove that $n^2=m^2$ if and only if $m=n$ or $m=-n$.

Soln: consider the propositions,

$p: n^2=m^2$, $q: m=n$, & $r: m=-n$.

we have to prove that $p \leftrightarrow (q \vee r)$ is true.

First assume, that $q \vee r$ is true, then $m=n$ or $m=-n$

so that $m^2=n^2$.

i.e, p is true, thus, $(q \vee r) \rightarrow p$ is true.

Next, assume that $\neg(q \vee r)$ is true; i.e, $(\neg q) \wedge (\neg r)$ is true.

Then q is false and r is false.

i.e, $m \neq n$ and $m \neq -n$.

Then $m^2 \neq n^2$; i.e, $\neg p$ is true. This proves that

$\neg(q \vee r) \rightarrow \neg p$ is true.

\therefore statement $p \rightarrow (q \vee r)$ is true.

Thus both of $(q \vee r) \rightarrow p$ & $p \rightarrow (q \vee r)$ are true.

Hence $p \leftrightarrow (q \vee r)$ is true.

proof by contradiction:

The Indirect method of proof is equivalent to proof by contradiction.

The lines of argument in this method of proof of the statement $p \rightarrow q$ are as follows:

1. Hypothesis: Assume that $p \rightarrow q$ is false. That is, assume that 'p' is true and 'q' is false.
2. Analysis: Starting with the hypothesis that 'q' is false and employing the rules of logic and other known facts, infer that 'p' is false. This contradicts the assumption that 'p' is true.
3. Conclusion: Because of the contradiction arrived in the analysis, we infer that $p \rightarrow q$ is true.

eg (8): provide a proof by contradiction of the following statement:
for every integer n , if n^2 is odd, then 'n' is odd.

Soln: Let n be any integer. Then the given statement reads
 $p \rightarrow q$, where

p : n^2 is odd and q : n is odd

Assume that $p \rightarrow q$ is false, i.e., assume 'p' is true and 'q' is false. Now q is false means: 'n' is even, so that
 $n = 2k$ for some integer k .

This yields $n^2 = (2k)^2 = 4k^2$ from which it is evident that
 n^2 is even. i.e., p is false.

This contradicts the assumption that 'p' is true.

In view of this, the given conditional $p \rightarrow q$ is true.

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Ex (9): prove the statement:

"The square of an even Integer is an even Integer"

By the method of contradiction.

Soln: Here, statement is in the form $p \rightarrow q$.

where, $p: n$ is an even Integer

$q: n^2$ is an even Integer

Assume $p \rightarrow q$ is false, i.e., assume 'p' is true & 'q' is false.

Since 'q' is false $\neg q$ is true. i.e., n^2 is not an even Integer.

$\therefore n^2 = n \times n$ is not divisible by 2.

This implies that 'n' is not divisible by 2. i.e., 'n' is not an even Integer. i.e., 'p' is false which contradicts the assumption 'p' is true.

\therefore The given proposition $p \rightarrow q$ is true

Ex (10) prove that if 'm' is an even Integer, then $m+7$ is an odd Integer.

Soln: Here, the given statement is $p \rightarrow q$ where,

$p: m$ is even, $q: m+7$ is odd.

Assume that $p \rightarrow q$ is false. i.e., assume 'p' is true & 'q' is false. Since 'q' is false, $m+7$ is even.

Hence, $m+7 = 2k$ for some Integer k, This yields,

$$m = 2k - 7 = (2k - 8) + 1 = 2(k - 4) + 1.$$

which shows that 'm' is odd. This means that 'p' is false, which contradicts the assumption that 'p' is true.

In view of contradiction, given statement $p \rightarrow q$ is true.

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Q (1): prove that for all real numbers x & y , if $x+y > 100$ then $x > 50$ or $y > 50$.

Soln: Take any two real numbers x and y . Then the statement to be proved reads $p \rightarrow (q \vee r)$ where,

$p \equiv p(x, y): x+y > 100$, $q \equiv q(x): x > 50$, $r \equiv r(y): y > 50$.

Assume that 'p' is true and $q \vee r$ is false. Since $q \vee r$ is false, q is false and r is false.

This means that $x < 50$ & $y < 50$. This yields $x+y < 100$. Thus 'p' is false. This contradicts the assumption that 'p' is true.

Hence the given statement $p \rightarrow q$ is true

Q (2): give (i) a direct proof, (ii) an indirect proof, (iii) proof by contradiction for the following statement:
"If 'n' is an odd integer, then $n+9$ is an even integer."

Soln: (i) Direct proof: Assume that 'n' is an odd integer, then $n = 2k+1$ for some integer k . This gives $n+9 = (2k+1)+9 = 2(k+5)$ from which $n+9$ is even.
Hence the direct proof

(ii) Indirect proof: Assume that $n+9$ is not an even integer. Then $n+9 = 2k+1$ for some integer k . This gives $n = (2k+1)-9 = 2(k-4)$, which shows that 'n' is even. Thus if $n+9$ is not even, then 'n' is not odd.

This proves the contrapositive of the given statement, which is a indirect proof.

(iii) proof by contradiction: Assume that the given statement is false. That is, assume that 'n' is odd & $n+9$ is odd, since $n+9$ is odd, $n+9 = 2k+1$ for some integer k , so that $n = (2k+1)-9 = 2(k-4)$ which shows that 'n' is even. This contradicts the assumption that 'n' is odd. Hence given statement is true //

proof by exhaustion:

Recall that a proposition of the form " $\forall x \in S, p(x)$ " is true if $p(x)$ is true for every x in S .

If ' S ' consists of only limited number of elements, we can prove that the statement " $\forall x \in S, p(x)$ " is true by considering $p(a)$ for each a in S & verifying that $p(a)$ is true. (in each case).

ex: (13): prove that every even integer ' n ' with $2 \leq n \leq 26$ can be written as a sum of at most 3 perfect squares.

soln: Let $S = \{2, 4, 6, \dots, 24, 26\}$ we have to prove that the statement: " $\forall x \in S, p(x)$ " is true where

$p(x)$: x is a sum of at most three perfect squares.

observe that the following:

$$2 = 1^2 + 1^2$$

$$4 = 2^2$$

$$6 = 2^2 + 1^2 + 1^2$$

$$8 = 2^2 + 2^2$$

$$10 = 3^2 + 1^2$$

$$12 = 2^2 + 2^2 + 2^2$$

$$14 = 3^2 + 2^2 + 1^2$$

$$16 = 4^2$$

$$18 = 4^2 + 1^2 + 1^2$$

$$20 = 4^2 + 2^2$$

$$22 = 3^2 + 3^2 + 2^2$$

$$24 = 4^2 + 2^2 + 2^2$$

$$26 = 5^2 + 1^2$$

The above facts verify that each ' x ' in S is a sum of at most 3 perfect squares

proof of existence:

It was pointed out that a proposition of the form " $\exists x \in S, p(x)$ " is true if any one element $a \in S$ such that $p(a)$ is true is exhibited.

Hence, the best way of proving a proposition of the form " $\exists x \in S, p(x)$ " is to exhibit the existence of one $a \in S$ such that $p(a)$ is true.

Q14: prove that there exists a real number x such that $x^3 + 2x^2 - 5x - 6 = 0$.

Soln: It is sufficient to exhibit one real number x such that $x^3 + 2x^2 - 5x - 6 = 0$.

\therefore If we put $x = -1$, then the result will be proved

Q15: prove that there exist positive Integers m and n such that m, n and $m+n$ are all perfect squares.

Soln: Note, that $m = 9$ & $n = 16$ are perfect squares & $m+n = 25$ which is a perfect square.

Disproof by contradiction:

Suppose we wish to disprove a conditional $p \rightarrow q$. For this, we start with the hypothesis that 'p' is true & q is true, & end up with a contradiction. In view of the contradiction, we conclude that the conditional $p \rightarrow q$ is false. This method of disproving $p \rightarrow q$ is called disproof by contradiction.

Q16: Disprove the statement:

"The sum of two odd Integers is an odd Integer"

Soln: The proposition to be prov. disproved is $p \rightarrow q$ where, p : a & b are odd Integers, & q : $a+b$ is an odd Integer.

Assume that 'p' is true & 'q' is true.

Then, $a = 2k_1 + 1$, $b = 2k_2 + 1$ --- (1)

$a+b = 2k_3 + 1$ --- (2)

for some Integers k_1, k_2, k_3

From (1) we get $a+b = 2(k_1+k_2+1)$ which shows that $a+b$ is an even Integer.

This contradicts that assumption (2).

\therefore This infers that $p \rightarrow q$ is false, which disproves given statement

Disproof by counter example:

Recall that a proposition of the form " $\forall x \in S, p(x)$ " is false if any one element $a \in S$, such that $p(a)$ is false is exhibited. Hence the best way of disproving a proposition involving the universal quantifier is to exhibit just one case where the proposition is false.

eg (17): Disprove the proposition: The product of any two odd integers is a perfect square.

Soln: we note that $m=3$ & $n=5$ are odd integers, but $mn=15$ is not a perfect square.

Thus the given proposition is disproved, with $m=3$ & $n=5$ as counter example

eg (18): prove or disprove that if m and n are positive integers which are perfect squares, ~~but $m \neq n$~~ then $m+n$ is a perfect square.

Soln: Note that $m=9$ and $n=4$ are perfect square, but $m+n=14$ is not a perfect square.

\therefore The given statement is not true.

It is disproved through the counter example $m=9, n=4$